STATIONARY SOLUTIONS OF THE EQUATIONS OF MOTION OF A LIQUID WITH GAS BUBBLES

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It is well known that a liquid with gas bubbles is an example of a nonlinear dispersive medium, and the existence of stationary perturbations in it is due to the mutual compensation of nonlinear and dispersion effects. In this case the nonlinearity of the gas–liquid mixture is determined by the hydrodynamic nonlinearity, the nonlinearity of bubble oscillations, and the equation of state of the liquid component of the mixture. The compressibility of the mixture depends on the compressibilities of its liquid and gas components. Stationary solutions of the system of equations were obtained in [1-3] for a two-phase medium, including a second-order nonlinear equation of the Rayleigh type for an isolated cavity. It was assumed in these papers that there is no bubble motion relative to the liquid, and that their number per unit volume of the mixture is constant. Besides, the hydrodynamics equations were linearized in [2], and the liquid component of the mixture was assumed incompressible in [1]. Stationary solutions of the full system of nonlinear equations of motion of a liquid with gas bubbles are obtained in the present paper with a single assumption: there is no bubble motion relative to the liquid. They are analyzed qualitatively, and the effect of simultaneous account of the compressibility of the liquid component of the medium and of hydrodynamic nonlinearities is explained.

Wave propation in a liquid with gas bubbles has been considered within the two-phase model suggested in [4, 5]. According to this model, the motion of a two-phase medium is described by the conservation equations of mass, momentum, number of bubbles, and energy accurately up to first order in the bulk concentration of the gas k. In the one-dimensional case these equations can be represented in the form

$$\begin{gathered}\partial\bar{\rho}/\partial t+\partial(\bar{\rho}u)/\partial x=0,\ \partial(\bar{\rho}u)/\partial t+\partial(p+\bar{\rho}u^2)/\partial x=0,\\ \partial N/\partial t+\partial(Nu)/\partial x=0,\ \bar{\rho}=(1-k)\rho,\ k=4\pi NR^3/3,\\ Rd^2R/dt^2+(3/2)(dR/dt)^2=[p_0(R_0/R)^{3\gamma}-p]/\rho(p_0),\end{gathered}\tag{1}$$

where $\bar{\rho}$, u, p are the medium density, velocity, and pressure; ρ, density of the liquid component of the mixture; p_0 and c_0, equilibrium values of the pressure and speed of sound in the pure liquid; R_0 and k_0, equilibrium values of the bubble radius and bulk concentration of the gas; R, varying bubble radius; N, number of bubbles per unit volume; γ, adiabatic index for the gas in the bubble; and $d/dt = \partial/\partial t + u\partial/\partial x$.

We introduce the dimensionless variables $t' = t\omega_0$, $x' = x\omega_0/c_0$, $u = u'c_0$, $p = p'\rho_0 c_0^2$, $p_0 = p_0'\rho_0 c_0^2$, $V = (R/R_0)^3$, $\bar{\rho} = \bar{\rho}'\rho_0$, $\omega_0^2 = 3\gamma p_0/\rho_0 R_0^2$, $\rho_0 = \rho(p_0)$ (in what follows the prime is omitted). In this case the Teta equation [6] acquires the form $p - p_0 = (\rho^n - 1)/n$, where n is the adiabatic index for the liquid. This approach makes it possible to obtain for n = 1 a linear equation of state, i.e., to consider the acoustic approximation. For n = 7.15 we have a nonlinear equation of state of the liquid. Due to the presence of nonlinear convective terms in it, the system of equations (1) is conveniently considered in Lagrangian coordinates. After transforming to the mass Lagrangian coordinate ξ, this system acquires the form

$$\partial\bar{\rho}/\partial t+\bar{\rho}^2\partial u/\partial\xi=0,\ \partial u/\partial t+\partial p/\partial\xi=0;\tag{2}$$

$$\partial k/\partial t+k\bar{\rho}\partial u/\partial\xi=(3k/R)\partial R/\partial t,\ \bar{\rho}=(1-k)\rho;\tag{3}$$

$$\partial^2V/\partial t^2-(V^{-1}/6)(\partial V/\partial t)^2=V^{1/3}(V^{-\gamma}-p/p_0)/\gamma.\tag{4}$$

We consider the stationary solutions of the system (2)-(4), i.e., solutions depending on $\eta = \xi - ct$, where c is a constant, equal to the velocity (in units of c_0) of displacement of some perturbation in the medium. Equations relating all unknown quantities with the bubble volume V follows from (2), (3) for the stationary solutions:

$$c(u-u_e)=p-p_e,\ k=m\rho V/(1+m\rho V);\tag{5}$$

$$p/c^2+(1+n(p-p_0))^{-1/n}=B-mV,\tag{6}$$

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where u_e, p_e, as well as k_e, V_e is the given state of the medium for some fixed value $\eta = \eta_e$. The values of the quantities m and B, calculated from the given state of the medium, equal $m = k_e/((1-k_e)V_e\rho_e)$, $B = p_e/c^2 + 1/\rho_e + k_e/((1-k_e)\rho_e)$, where $\rho_e = (1 + n(p_e - p_0))^{1/n}$. We point out that for a fixed solution m and B are constants, i.e., the values of these quantities, calculated from the state of the medium at arbitrary points of the wave corresponding to this solution – coincide. For the stationary solutions relations (5), (6) determine the set of possible states of the medium into which the given state can evolve.

The function V(p), defined by Eq. (6), has a maximum V_m at $p = p_m$, where $p_m = p_0 + (c^{2n/(n+1)} - 1)/n$, while $dV/dp < 0$ for $p > p_m$, and $dV/dp > 0$ for $p < p_m$. The inverse function p(V) is defined for $0 < V \leq V_m$, and has two branches: a right one and a left one, where, respectively, $p \geq p_m$ and $p \leq p_m$. The quantity p_e determines the branch of the function p(V), on which the given state of the medium is found. For the linear equation of state of the liquid (n = 1) this function has the explicit shape

$$p(V) = p_0 + (\rho_e + c^2/\rho_e - c^2m(V - V_e) \pm \sqrt{(\rho_e + c^2/\rho_e - c^2m(V - V_e))^2 - 4c^2} - 2)/2,$$

where the right branch corresponds to the plus sign, and the left one – to the minus sign. We compare the nonlinear dependence of V(p) (6), corresponding to the acoustic approximation for the liquid component of the medium, with the similar dependences obtained in [1-3]. The hydrodynamic equations in [1] are nonlinear, and the liquid component of the medium is incompressible. In this case the relation between V and p is linear, while for all c values $dV/dp < 0$. If the hydrodynamic equations are linear, but the liquid component of the medium is compressible, then, as shown in [2], the relation between V and p is linear too. The pressure dependence of the bubble volume, similar to that obtained in [2], follows from the linear equation (2), and is

$$V - V_e = (p - p_e)(c^2 - 1)/c^2m. \tag{7}$$

In the p, V plane the straight line (7) is tangent to the curve (6) at the point (p_e, V_e) only for $p_e = p_0$. For other p_e values they intersect at this point. Let $c^2 > 1$. Then for $p_e < p_m$ ($c_e^2 < c^2$, $c_e^2 = (1 + n(p_e - p_0))^{(n+1)/n}$) the signs of the derivatives dV/dp of the dependences (6), (7) coincide for $p = p_e$. If $p_e \geq p_m$ ($c^2 \leq c_e^2$), the signs of these derivatives are different, i.e., unlike (7), the bubble volume decreases with increasing pressure in (6). A similar distinction in the behavior of the bubble with varying pressure also occurs for $c^2 < 1$ at $p_e < p_m$. For $c^2 = 1$ there exists no pressure dependence of the bubble volume (7), as well as of stationary solutions differing from the trivial one, for which $\dot{V} = dV/d\eta = 0$. In the nonlinear case stationary solutions can also exist for $c^2 = 1$. Unlike [1, 2], for a fixed velocity value c the sign of the derivative dV/dp in (6) can vary, i.e., if there exists a stationary solution in which p acquires values larger and smaller than p_m, for this solution the bubble volume can both increase and decrease with increasing pressure, depending on whether $p < p_m$ or $p > p_m$. Besides, unlike [2], the restriction on the magnitude of the bubble volume $V \leq V_m$ does not follow from the smallness condition of the acoustic Mach number, but is a consequence of the nonlinearity of the hydrodynamic equations and of the compressibility of the liquid component of the medium. Thus, comparison of (6) with the similar dependences V(p), obtained in [1, 2], shows that account of hydrodynamic nonlinearities and of the compressibility of the liquid can lead not only to quantitative, but also to qualitative changes in the stationary solutions. We note that a relation between V and p was obtained in [3] for stationary solutions with account of hydrodynamic nonlinearities and liquid compressibility, but, as in [2], it is linear and has the form (7).

We introduce the quantities $V' = V/V_e$, $m' = mV_e$ (the primes are omitted again). The first integral for $V(\eta)$ follows from Eq. (4)

$$V^{-1/3}\dot{V}^2 = \beta^2(U(V) + H), \tag{8}$$

where

$$U(V) = -mp_0(V_e^{-\gamma}(V^{1-\gamma} - 1) + (\gamma - 1)(V - 1))/(\gamma - 1) + (p - p_e) \times$$
$$\times (p + p_e - 2p_0)/2c^2 - [(p - p_0 + 1)(1 + n(p - p_0))^{-1/n} - (p_e - p_0 + 1)(1 + n(p_e - p_0))^{-1/n}]/(n - 1);$$
$$\beta^2 = 2/c^2m\gamma p_0V_e^{2/3};\ H = \dot{V}_e^2/\beta^2;\ \dot{V}_e = dV/d\eta \text{ for } \eta = \eta_e;$$

and p(V) is defined by relation (6) (for n = 1 the last term of U(V) acquires the form $[\ln(1 + p - p_0) + 1/(1 + p - p_0) - \ln(1 + p_e - p_0) - 1/(1 + p_e - p_0)]$). Thus, the study of wave stabilization in a liquid with gas bubbles reduces to a study of the solution of Eq. (8). We explain the existence of a bounded solution of this equation for a fixed c value and a given state of the medium p_e, V_e, k_e, $\dot{V}_e$. The right-hand side of Eq. (8) describes a family of curves depending on the parameters m, p_e, V_e, c, H. A solution of (8) may exist for those portions of the curves for which the quantity U(V) + H is nonnegative. This set of curves will be investigated by quali-

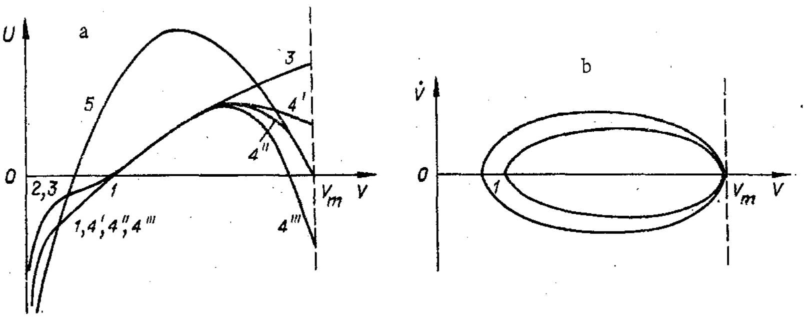

Fig. 1

tative methods. For this it is necessary to know the shape of the function $U(V)$ on the interval $(0, V_m]$ as a function of the parameters p_e, V_e, m, c. This function, whose derivative is

$$U'(V) = m\left(p_0 V_e^{-\gamma} V^{-\gamma} - p(V)\right), \tag{9}$$

has the following properties on the interval under consideration: $U(1) = 0$, $U''(V) < 0$ for the left branch of $p(V)$, while for the right branch $U''(V)$ increases monotonically from $-\infty$ to $+\infty$ with increasing V. We find the conditions for which there is a transition from one branch of the function $p(V)$ to the other at the point (p_m, V_m) of the p, V plane. Unlike the tendency to a simple root, the nonconstant solution of Eq. (8) tends to a double root of the function $U(V) + H$ only asymptotically for $\eta \to \infty$ or for $\eta \to -\infty$. For the maximum value of the bubble radius $V_m = V(\eta_m)$ to be equal to the exact value of the nonconstant integral (8) it is necessary that $U(V_m) + H = 0$. Expanding the indeterminate form $(U(V_m) + H)/(dV(p_m)/dp)^2$, we find that $V^{-1/3}\dot{p}^2$ tends to the quantity $-\beta^2 mc^{2(2n+1)/(n+1)}\, U'(V_m)/(2n+2)$ vith $\eta \to \eta_m$ $(V \to V_m,\ p \to p_m)$. Therefore, if V_m is a simple root $(U'(V_m) < 0)$, at the moment of time $\eta = \eta_m$ there is a transition from one branch of the function $p(V)$ to the other at the point (p_m, V_m) on the p, V variable plane. At the same time, $\ddot{V}(\eta)$ is continuous at $\eta = \eta_m$, since the limits of $U'(V)$ when V tends to V_m from the left and right branches coincide. If V_m is a double root $(U'(V_m) = 0)$, then $V \to V_m$, $p \to p_m$, $\dot{p} \to 0$ asymptotically for $\eta \to \eta_m$, $\eta_m = \infty$ or $\eta_m = -\infty$.

Obviously, $U'(1) > 0$ for $p_e \le 0$, $0 < V_e < \infty$. Besides, for any fixed value of $p_e > 0$ there exists a unique $\overline{V}_e$, such that $U'(1) > 0$ for $V_e < \overline{V}_e$, $U'(1) = 0$ for $V_e = \overline{V}_e$, and $U'(1) < 0$ for $V_e > \overline{V}_e$. To study the function $U(V)$ it is convenient to divide the whole set of parameters p_e, V_e into subsets: $p_e \le 0$, $0 < V_e < \infty$; $p_e > 0$, $V_e > \overline{V}_e$; $p_e > 0$, $V_e = \overline{V}_e$; $p_e > 0$, $V_e < \overline{V}_e$. In this case the fixed k_e value is arbitrary within the model under consideration. The branch of the funtcion $p(V)$, corresponding to a given state of the medium, is determined by the relation between the quantities c^2 and c_e^2. Therefore, for given values of p_e, V_e we study the qualitative nature of the function $U(V)$ in three cases, corresponding to the following variations: $c^2: c^2 < c_e^2$, $c^2 = c_e^2$, $c^2 > c_e^2$. We turn now to consider the sets mentioned above of the parameters p_e, V_e.

Let $p_e \le 0$, $0 < V_e < \infty$, i.e., $U'(1) > 0$. Using the properties of $U(V)$, we verify that for the set of p_e, V_e considered the qualitative nature of this function is as represented on Fig. 1a. Curves 1, 2 correspond to the left and right branches of $p(V)$ for $c^2 = c_e^2$ for $0 < V \le 1$, $V_m = 1$, since for $c^2 < c_e^2$ for $1 \le V \le V_m U'(V) > 0$, $p_m \le p \le p_e$ we have the obvious inequality $p_0 V_e^{-\gamma} V^{-\gamma} - p > 0$. The curve of $U(V)$ for $c^2 < c_e^2$ is shown on Fig. 3. For $c^2 > c_e^2$ the shape of $U(V)$ as a function of the c value is shown by curves 4', 4'', 4'''. For $c^2 > c_e^2$ the relation between V and p is realized by the left branch of $p(V)$, for which $U''(V) < 0$. Therefore, for the existence of a solution (8) it is sufficient that $U_1(c^2) = U(V_m(c^2)) \le 0$. The shape of this function, whose derivative is $dU_1/dc^2 = (c_e^{2n/(n+1)} - c^{2n/(n+1)})[2p_0(1 - V_e^{-\gamma} V_m^{-\gamma}) + (c_e^{2n/(n+1)} + c^{2n/(n+1)} - 2)/n]/2nc^4$, is illustrated schematically on Fig. 2a. It can be shown that for a given state p_e, V_e, k_e there exists a unique value $c_1^2 > (1 - np_0)^{(n+1)/n}$ such that for $c^2 \geqslant c_1^2$ the inequality $(U(V_m(c^2)) \le 0$. Thus, for $c^2 < c_1^2$ there exists no solution of Eq. (8) for the quantities under consideration p_e, V_e, k_e and arbitrary $\dot{V}_e(H)$ values. For an arbitrary fixed value $c^2 \ge c_1^2$ the range of admissible H values, for which a solution of Eq. (8) exists for given p_e, V_e, k_e, is determined by the inequality $0 \le H \le -U_1(c^2)$. For $0 \le H < -U_1(c^2)$ the solution of (8) is periodic, with the relation between V and p realized by the left branch of $p(V)$. The qualitative nature of this solution is shown on Fig. 2b. At $H = -U_1(c^2)$, due to the fact that $U'(V_m) < 0$ there is a transition from one branch of $p(V)$ to the other on the p, V variable plane. In this case there occurs a transition in the phase plane (see Fig. 1b) at the point $V = V_m$, $\dot{V} = 0$ from one phase trajectory, corresponding to one of the branches of $p(V)$, to another phase trajectory, corresponding to the other branch. A graph of $U(V)$, for which the relation between V and p is realized by the right branch of $p(V)$, is shown on Fig. 1a, curve 5. For $H = -U_1(c^2)$ the solution is also periodic. The shape of this solution is shown on Fig. 2c. Unlike the solution shown on Fig. 2b, it is characteristic of this

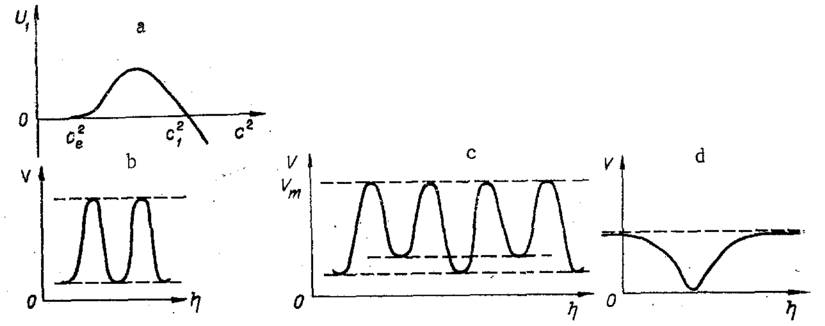

Fig. 2

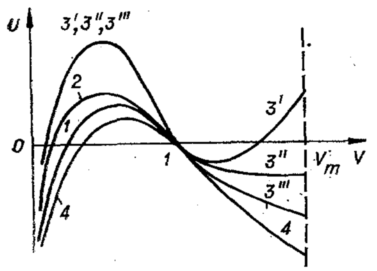

Fig. 3

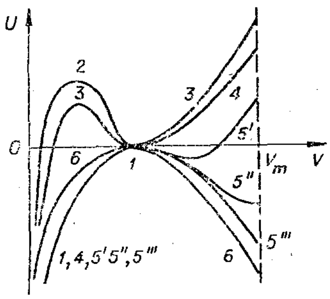

Fig. 4

solution that bubble collapse and expansion occur twice during a period. One of the collapses occurs at the moment of maximum pressure in the medium, and the other – during the dilatation phase due to the inertia properties of associated mass of the bubble. Thus, for $p_e \leq 0$, $0 < V_e < \infty$ and arbitrary values of k_e, $\dot{V}_e$ no solution of (8) exists for $c^2 \leq (1 - np_0)^{(n+1)/n}$. It hence follows that for stationary solutions, whose velocities are $c^2 \leq (1 - np_0)^{(n+1)/n}$, the pressure satisfies the inequality $p - p_0 > -p_0$, the minimum sound pressure in the medium, by which stationary perturbations propagate with velocities of the range mentioned, being p_0.

Let $p_e > 0$, $V_e > \overline{V}_e$, i.e., $U'(1) < 0$. For the given set of parameters p_e, V_e the shape of $U(V)$ is shown on Fig. 3. Curves 1, 2 correspond to the left and right branches of $p(V)$ for $c^2 = c_e^2$. The solution for this velocity value exists for $H = 0$, where in this case $V_m = 1$. Its shape agrees qualitatively with the solution illustrated on Fig. 2c. For $c^2 < c_e^2$ the shape of $U(V)$ is shown as a function of the c value by curves 3', 3'', 3'''. It is seen from the properties of the function $U(V)$ that for all c for which $U'(V_m(c^2)) \geq 0$ (curves 3', 3'') there exists a single value $V_1 \in (1, V_m]$, such that $U'(V_1) = 0$. For these c values with $0 \leq H < -U(V_1)$ the solution of (8) is periodic, qualitatively agreeing with that shown on Fig. 2b. For $H = -U(V_1)$ the solution has the form of a soliton (Fig. 2d). For $H > -U(V_1)$ Eq. (8) has no solution for the c values mentioned. For c values for which $U'(V_m(c^2)) < 0$ (curve 3'''), for $0 \leq H < -U(V_m)$ the solution is periodic, qualitatively in agreement with that illustrated on Fig. 2b (Fig. 2c). For $c^2 > c_e^2$ the shape of $U(V)$ is shown by curve 4. For these c values at $0 \leq H < -U(V_m)$ $(H = -U(V_m))$ the solution is periodic, and its schematic shape is given on Fig. 2b (Fig. 2c).

Let $p_e > 0$, $V_e = \overline{V}_e$, i.e., $U'(1) = 0$. For this set of parameters p_e, V_e the shape of $U(V)$ is given on Fig. 4. We note that for the p_e, V_e values under consideration the right-hand side of Eq. (4) vanishes. Curves 1, 2 correspond to the left and right branches of $p(V)$ for $c^2 = c_e^2$. The solution corresponding to the right branch is a soliton, qualitatively in agreement with that illustrated on Fig. 2d. In this case $\eta_e = \infty$ or $\eta_e = -\infty$, $H = 0$. There exists no solution corresponding to the left branch. For $c^2 < c_e^2$ the shape of $U(V)$ as a function of the c value is shown by curves 3, 4, 5', 5'', 5'''. For $c_*^2 < c^2 < c_e^2$, $c_*^2 = \gamma p_e c_e^2/(\gamma p_e + m c_e^2)$ $(U''(1) > 0$, curve 3) the solution is a soliton, with $H = 0$, $\eta_e = \infty$ or $\eta_e = -\infty$. For $c^2 = c_*^2$ $(U''(1) = 0$, curve 4) there exists no solution. For $c^2 < c_*^2 (U''(1) < 0$, curves 5', 5'', 5''') a solution exists for $H > 0$. Further study for $c^2 < c_*^2$ is performed similarly to the case $p_e > 0$, $V_e > \overline{V}_e$ (curves 3', 3'', 3''' on Fig. 3). For $c^2 > c_e^2$ (curve 6) the solution for $0 < H < -U(V_m)$ $(H = -U(V_m))$ is periodic, in qualitative agreement with the one illustrated on Fig. 2b (Fig. 2c).

Let $p_e > 0$, $V_e < \overline{V}_e$, i.e., $U'(1) > 0$ $(V_e^{\gamma} p_e/p_0 < 1)$. The study of existence of solutions for $c^2 \geq c_e^2$ is carried out similarly to the case $p_e \leq 0$ (curves 1, 2, 4', 4'', 4''' of Fig. 1a). For $c^2 < c_e^2$ the shape of $U(V)$ is shown on Fig. 5a. The inequality $U'(V(p)) \geqslant 0$ is equivalent to the inequality

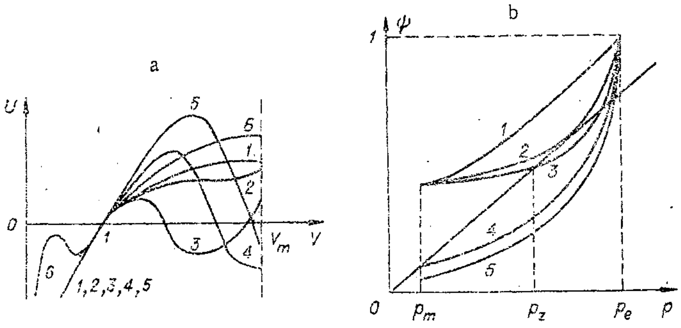

Fig. 5

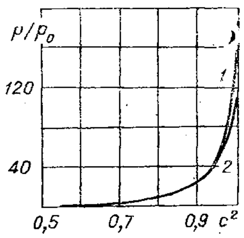

Fig. 6

$$pV_e^{\gamma}/p_0 \leqslant \psi(p), \tag{10}$$

where $\psi(p) = (1 + ((p_e - p)/c^2 + (1 + n(p_e - p_0))^{-1/n} - (1 + n(p - p_0))^{-1/n})/m)^{-\gamma}$, and the function $\psi(p)$ increases monotonically on the interval $[p_m, p_e]$. Besides, $\psi(p_e) = 1$, $\psi'(p_m) = 0$, $\psi''(p) > 0$. If $\psi'(p_e) \leq V_e^2/p_0$, then inequality (10) is valid for all $p \in [p_m, p_e]$. The inequality $\psi'(p_e) \leq V_e^{\gamma}/p_0$ holds for $c^2 \geq c_2^2$, $c_2^2 = c_e^2/(1 + mc_e^2V_e^{\gamma}/\gamma p_0)$. For these velocity values the shape of $U(V)$ is shown by curve 6 on Fig. 5a. For the p_e, V_e values considered there exists no solution for $c_2^2 \leq c^2 < c_e^2$. Let $c^2 < c_2^2$, i.e., $\psi'(p_e) > V_e^{\gamma}/p_0$. For a fixed c value the derivative $U'(V_m(c^2))$ can be larger than, equal, or smaller than zero. We consider each of these cases. Let $U'(V_m(c^2)) > 0$ ($\psi(p_m) > p_mV_e^{\gamma}/p_0$). In this case it follows from the properties of the function $\psi(p)$ that there exists a unique value $p_\xi \in [p_m, p_e]$ such that $\psi'(p_\xi) = V_e^{\gamma}/p_0$. The shape of $\psi(p)$ as a function of values of p_e, V_e, k_e is shown on Fig. 5b by curves 1-3. Which of these curves corresponds to a given state of the medium is determined by the relation between $\psi(p_\xi)$ and $V_e^{\gamma}p_\xi/p_0$. If $\psi(p_\xi) \geq V_e^{\gamma}p_\xi/p_0$, curves 1, 2, which lie above the straight line of pV_e^{γ}/p_0 hold. In that case $U'(V(p)) \geq 0$ on $[1, V_m]$, and the shape of $U(V)$ corresponds to curves 1, 2 on Fig. 5a. There exists no solution in this case. If $\psi(p_\xi) < V_e^{\gamma}p_\xi/p_0$, curve 3 (Fig. 5b) holds. In this case, for the existence of a solution it is necessary to determine the sign of $U(V(p_z))$ (Fig. 5b). If $U(V(p_z)) \leq 0$, then for $0 \leq H < -U(V(p_z))$ the solution is periodic (Fig. 2b), while for $H = -U(V(p_z))$ the solution has the shape of a soliton (Fig. 2d). In this case the shape of $U(V)$ is shown on curve 3 of Fig. 5a. Let $U(V(p_z)) > 0$. In this case the solution of Eq. (8) does not exist. Let $U'(V_m(c^2)) = 0$, ($\psi(p_m) = p_mV_e^{\gamma}/p_0$). The shapes of $U(V)$ and $\psi(p)$ are shown for this value on Figs. 5a, b, respectively, by curves 4. For solutions to exist it is sufficient that $U'(V_m(c^2)) \leq 0$. For $0 \leq H < -U(V_m)$ the solution is periodic (Fig. 2b), while for $H = -U(V_m)$ a soliton occurs (Fig. 2d). Let $U'(V_m(c^2)) < 0$ ($\psi(p_m) < p_mV_e^{\gamma}/p_0$. For these c values the shapes of the functions $U(V)$ and $\psi(p)$ are shown by curves 5, respectively, on Figs. 5a, b. For the existence of solutions it is sufficient that $U(V_m(c^2)) < 0$. In that case, for $0 \leq H > -U(V_m)$ ($H = -U(V_m)$) the solution is in qualitative agreement with that illustrated on Fig. 2b, c.

Thus, one can determine the existence and qualitative behavior of a stationary perturbation with an arbitrary fixed velocity for a given state of the medium. In this case the solution of (8) for given p_e, V_e, k_e, $\dot{V}_e$, and c for which it exists is written down implicitly:

$$\beta\eta - \beta\eta_e = \pm \int_1^V y^{-1/6}(U(y) + H)^{-1/2}dy.$$

In conclusion, it is necessary to point out the features of the solution, related to nonlinearity effects of the equation of state of the liquid component of the medium. In the p, V variable plane the curves (6) corresponding to the linear and nonlinear equation of state of the liquid do not coincide. In particular, the p_m values are different for $c^2 \neq 1$. The tangents to these curves at the point (p_e, V_e) are $m(V - V_e) = ((1 + p_e - p_0)^{-2} - c^{-2})(p - p_e)$, $m(V - V_e) = ((1 + n(p_e - p_0))^{-(n+1)/n} - c^{-2})(p - p_e)$. It is seen that for $p_e \neq p_0$ the slopes of the tangents are different. This difference is most important for $c^2 = 1$. Therefore, the solutions obtained with and without account of the nonlinearities of the equation of state of the liquid can differ from each other. Thus, Fig. 6 shows the dependence of the soliton amplitude on the square of its velocity ($c_*^2 < c^2 \leq c_e^2$). Curve 1 corresponds to a linear equation of state of the liquid, and curve 2 – to the nonlinear. The equilibrium state of the medium at $\eta_e = \pm\infty$ is of the form $p_e = 2p_0$, $V_e^{-\gamma} = p_e/p_0 = 2$, $k_e = 10^{-4}$, $\dot{V}_e = 0$, $\gamma = 1, 4$, $p_0 = 10^5$ Pa$/\rho_0 c_0^2$. It is seen that for solitons whose velocities squared are smaller than 0.9 the amplitude coincide for the linear and nonlinear equations of state of the liquid. For $c^2 > 0.9$ the amplitudes differ substantially.

Thus, the exact solution of the nonlinear equations of motion of a liquid with gas bubbles has been obtained for one-dimensional stationary perturbations. In this case account of the hydrodynamic nonlinearity and of the compressibility of the liquid component of the medium leads to an extended class of stationary solutions.

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PROBLEM OF NONSTATIONARY TRANSPORT PHENOMENA IN MULTIPHASE MEDIA

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Nonstationary transport phenomena in multiphase medium in many ways are determined by kinetic processes at the interfaces. The simplest idealizations, introduced during Fourier's and Fick's times, when interphase kinetics were given by the boundary conditions of the type

$$\partial n_i/\partial R = a_{ij}(n_i - n_j),$$

cannot reflect the basic features of transport processes when the physical conditions at the interfaces change considerably and rapidly. This especially concerns problems with mobile boundaries, arising, for example, in analyzing the kinetics of phase transformation [1-5]. In the spherical variant, nonstationary effects arise, in particular, due to Laplacian pressure, which is clearly related to the motion of the boundary ($\sim 1/R(t)$).

We shall give a derivation of the general type of boundary kinetics, based on the process of one-dimensional transport of a fixed component of matter through the interface R of two media (phases), which is the surface of discontinuity for the concentration field of the given component. We shall examine the model indicated schematically in Fig. 1. It assumes that the volume of the media can be separated into some elementary regions of molecular size a_i and, in addition, they can vary in time kinetically and deformationally, i.e., $a_i = a_i(t)$. For solid media, the parameter a_i corresponds to a constant lattice, while for gas media it corresponds to the free path of particles. We assume that the motion of particles occurs in some potential field, whose average relief is shown schematically in Fig. 1. The presence of external and internal fields introduces an asymmetry into the potential relief of the particles, changing the kinetics of their transfer in the forward and backward directions. In what follows, the average velocities of such random wandering W_i will be distinguished

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